INTEGRATION
&
DIFFERENTIAL EQUATIONS

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INTEGRATION

INTEGRATION AS INVERSE PROCESS OF DIFFERENTIATION

Integration is the process of inverse differentiation .The branch of calculus which studies about Integration and its applications is called Integral Calculus.

Let $F(x)$ and $f(x)$ be two real valued functions of x such that,

$$\frac{d}{dx}F(x) = f(x)$$

Then, $F(x)$ is said to be an anti-derivative (or integral) of $f(x)$.
Symbolically we write $\int f(x)\,dx = F(x)$.

The symbol $\int$ denotes the operation of integration and called the integral sign.
$'dx'$ denotes the fact that the Integration is to be performed with respect to x .The function $f(x)$ is called the Integrand.

INDEFINITE INTEGRAL

Let $F(x)$be an anti-derivative of $f(x)$.
Then, for any constant 'C',

$$\frac{d}{dx}\{F(x) + C\} = \frac{d}{dx}F(x) = f(x)$$

So, $F(x) + C$ is also an anti-derivative of $f(x)$, where C is any arbitrary constant. Then, $F(x) + C$ denotes the family of all anti-derivatives of $f(x)$, where C is an indefinite constant.
Therefore, $F(x) + C$ is called the Indefinite Integral of $f(x)$.
Symbolically we write

$$\int f(x)\,dx = F(x) + C,$$

Where the constant C is called the constant of integration. The function $f(x)$ is called the Integrand.

Example :-Evaluate $\int \cos x \; dx$.
Solution:-We know that

$$\frac{d}{dx}\sin x = \cos x$$

So, $$\int \cos x \; dx = \sin x + C$$

ALGEBRA OF INTEGRALS

1. $\int [f(x) + g(x)]\,dx = \int f(x)dx + \int g(x)dx$

2. $\int k\, f(x)dx = k \int f(x)dx,$ for any constant k.

3. $\int [a\, f(x) + b\, g(x)]\,dx = a\int f(x)dx + b\int g(x)dx,$
for any constant a & b

INTEGRATION OF STANDARD FUNCTIONS

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C \;, (n \neq -1)$
2. $\int \frac{1}{x} dx = \ln|x| + C$
3. $\int cosx\, dx = \sin x + C$
4. $\int \sin x\, dx = -\cos x + C$
5. $\int sec^2 x\, dx = \tan x + C$
6. $\int cosec^2 x\, dx = -\cot x + C$
7. $\int \sec x \tan x\, dx = \sec x + C$
8. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$
9. $\int e^x dx = e^x + C$
10. $\int a^x dx = \frac{a^x}{\ln a} + C \;, (a > 0)$
11. $\int \tan x\, dx = \ln|\sec x| + C = -\ln|\cos x| + C$
12. $\int \cot x\, dx = \ln|\sin x| + C = -\ln|\operatorname{cosec} x| + C$
13. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$
14. $\int \operatorname{cosec} x\, dx = \ln|\operatorname{cosec} x - \cot x| + C$
15. $\int \frac{1}{\sqrt{1-x^2}} dx = sin^{-1} x + C$
16. $\int \frac{1}{1+x^2} dx = tan^{-1} x + C$
17. $\int \frac{1}{x\sqrt{x^2-1}} dx = sec^{-1} x + C$
18. $\int \frac{1}{\sqrt{x^2+1}} dx = \ln\left|x + \sqrt{x^2+1}\right| + C$
19. $\int \frac{1}{\sqrt{x^2-1}} dx = \ln\left|x + \sqrt{x^2-1}\right| + C$

Example:- Evaluate $\int \frac{a^2 sin^2 x + b^2 cos^2 x}{sin^2 2x} dx$

Solution:- $\int \frac{a^2 sin^2 x + b^2 cos^2 x}{sin^2 2x} dx$

$= \int \frac{a^2 sin^2 x + b^2 cos^2 x}{4 sin^2 x . cos^2 x} dx$

$= \frac{a^2}{4} \int \frac{1}{cos^2 x} dx + \frac{b^2}{4} \int \frac{1}{sin^2 x} dx$

$= \frac{a^2}{4} \int sec^2 x\, dx + \frac{b^2}{4} \int cosec^2 x\, dx$

$= \frac{1}{4} [a^2 \tan x - b^2 \cot x] + C$

INTEGRATION BY SUBSTITUTION

When the integrand is not in a standard form, it can sometimes be transformed to integrable form by a suitable substitution.

The integral $\int f\{g(x)\} g'(x) dx$ can be converted to
$\int f(t) dt$ by substituting $g(x)$ by t.

So that, if $\int f(t)dt = F(t) + C$,then

$\int f\{g(x)\}g'(x)dx = F\{g(x)\} + C.$

This is a direct consequence of CHAIN RULE.
For,

$$\frac{d}{dx}[F\{g(x)\} + C] = \frac{d}{dt}[F(t) + C].\frac{dt}{dx} = f(t).\frac{dt}{dx} = f\{g(x)\}g'(x)$$

There is no fixed formula for substitution.

Example:- Evaluate $\int \cos(2 - 7x)\, dx$
Solution:- Put $t = 2 - 7x$
So that $\frac{dt}{dx} = -7 \Rightarrow dt = -7dx$

$$\therefore \quad \int \cos(2 - 7x)\, dx = \frac{-1}{7}\int \cos t\, dt$$
$$= \frac{-1}{7}\sin t + C$$
$$= \frac{-1}{7}\sin(2 - 7x) + C$$

INTEGRATION BY DECOMPOSITION OF INTEGRAND

If the integrand is of the form $\sin mx \cdot \cos nx$, $\cos mx \cdot \cos nx$ or $\sin mx \cdot \sin nx$, then we can decompose it as follows;

1. $\sin mx \cdot \cos nx = \frac{1}{2} \cdot 2\sin mx \cdot \cos nx = \frac{1}{2}[\sin(m + n)x + \sin(m - n)x]$
2. $\cos mx \cdot \cos nx = \frac{1}{2}[\cos(m - n)x + \cos(m + n)x]$
3. $\sin mx \cdot \sin nx = \frac{1}{2}[\cos(m - n)x - \cos(m + n)x]$

Similarly, in many cases the integrand can be decomposed into simpler form, which can be easily integrated.
Example:-Integrate $\int \sin 5x \cdot \cos 2x\, dx$

Solution:- $\int \sin 5x \cdot \cos 2x\, dx$

$$= \frac{1}{2}\int[\sin(5 + 2)x + \sin(5 - 2)x]\, dx$$
$$= \frac{1}{2}\int(\sin 7x + \sin 3x)\, dx$$
$$= \frac{1}{2}\left[-\frac{1}{7}\cos 7x - \frac{1}{3}\cos 3x\right] + C$$
$$= -\frac{1}{14}\cos 7x - \frac{1}{6}\cos 3x + C$$

Example:-Integrate $\int \frac{\sin 6x + \sin 4x}{\cos 6x + \cos 4x}dx$

Solution:- $\int \frac{\sin 6x + \sin 4x}{\cos 6x + \cos 4x}dx = \int \frac{2\sin 5x\cos x}{2\cos 5x\cos x}dx$

$$= \int \frac{\sin 5x}{\cos 5x}dx$$

Put t = cos 5x, so that $\frac{dt}{dx} = -5\, sin5x \Rightarrow dt = -5sin5x.dx$

$$\therefore \int \frac{\sin 6x + \sin 4x}{\cos 6x + \cos 4x}dx = -\frac{1}{5}\int\frac{dt}{t} = -\frac{1}{5}ln|t| + C$$

$$= -\frac{1}{5}\ln|\cos 5x| + C$$
$$= \frac{1}{5}\ln|\sec 5x| + C$$

INTEGRATION BY PARTS

This rule is used to integrate the product of two functions.
If u and v are two differentiable functions of x, then according to this rule have;

$$\int uv\, dx = u\int v\, dx - \int \left[\frac{du}{dx}\int v\, dx\right] dx$$

In words, Integral of the product of two functions

$$= \boldsymbol{first\ function} \times (\boldsymbol{Integral\ of\ second\ function}) - \boldsymbol{Integral\ of}(\boldsymbol{derivative\ of\ first} \times \boldsymbol{Integral\ of\ second})$$

The rule has been applied with a proper choice of '$\boldsymbol{First}$' and '$\boldsymbol{Second}$' functions .Usually from among exponential function(**E**),trigonometric function(**T**),algebraic function(**A**),Logarithmic function(**L**) and inverse trigonometric function(**I**),the choice of '$\boldsymbol{First}$' and '$\boldsymbol{Second}$' function is made in the order of **ILATE.**

Example:-Evaluate$\int x\sin x\, dx$

Solution:-$\int x\sin x\, dx$

$$= x\int \sin x\, dx - \int \left[\frac{dx}{dx}.\int \sin x\, dx\right] dx$$
$$= -x\cos x + \int \cos x\ dx$$
$$= \sin x - x\cos x + C$$

Example:-Evaluate$\int e^x \cos 2x\, dx$

Solution:- $\int e^x\cos 2x\, dx = e^x\cos 2x - \int e^x(-2\sin 2x)\, dx$

$$= e^x\cos 2x + 2\int e^x \sin 2x\, dx$$
$$= e^x\cos 2x + 2\,[e^x\sin 2x - 2\int e^x\cos 2x\, dx]$$
$$= e^x\cos 2x + 2\,e^x\sin 2x - 4\int e^x\cos 2x\, dx + K$$

So,$5\int e^x\cos 2x = e^x[\cos 2x + 2\sin 2x] + K$

$\therefore \int e^x\cos 2x\, dx = \frac{e^x}{5}[\cos 2x + 2\sin 2x] + C$ (where $= {}^{K}/_{2}$)

INTEGRATION BY TRIGONOMETRIC SUBSTITUTION

The irrational forms $\sqrt{a^2 - x^2}$, $\sqrt{x^2 + a^2}$, $\sqrt{x^2 - a^2}$ can be simplified to radical free functions as integrand by putting $x = a\sin\theta$, $x = a\tan\theta$, $x = a\sec\theta$ respectively.
The substitution $x = a\tan\theta$ can be used in case of presence of $x^2 + a^2$ in the integrand, particularly when it is present in the denominator.

ESTABLISHMENT OF STANDARD FORMULAE

1. $\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\frac{x}{a} + C$

2. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} tan^{-1}\frac{x}{a} + C$

3. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} sec^{-1}\frac{x}{a} + C$

4. $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x + \sqrt{x^2 + a^2}\right| + C$

5. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right| + C$

Solutions:

1. Let $x = a\sin\theta$, so that $dx = a\cos\theta\, d\theta$ and $\theta = sin^{-1}\frac{x}{a}$

$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2sin^2\theta}} = \int \frac{a\cos\theta}{a\cos\theta} d\theta = \int d\theta = \theta + C = sin^{-1}\frac{x}{a} + C$

2. Let $x = a\tan\theta$, so that $dx = a\sec^2\theta\ d\theta$ and $\theta = tan^{-1}\frac{x}{a}$

$\therefore \int \frac{dx}{x^2+a^2} = \int \frac{a\sec^2\theta\ d\theta}{a^2\tan^2\theta + a^2} = \int \frac{a\sec^2\theta\ d\theta}{a^2(\tan^2\theta + 1)} = \int \frac{asec^2\theta}{a^2sec^2\theta} d\theta = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C$

$= \frac{1}{a} tan^{-1}\frac{x}{a} + C$

3. Let $x = a\sec\theta$, so that $dx = a\sec\theta\tan\theta\ d\theta$ and $\theta = sec^{-1}\frac{x}{a}$

$\therefore \int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta\ d\theta}{a\sec\theta\sqrt{a^2sec^2\theta-a^2}} = \int \frac{a\sec\theta\tan\theta}{a\sec\theta\, a\tan\theta} d\theta = \frac{1}{a}\int d\theta$

$= \frac{1}{a}\theta + C = \frac{1}{a} sec^{-1}\frac{x}{a} + C$

4. Let $x = a\tan\theta$, so that $dx = a\sec^2\theta\ d\theta$.

$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta\ d\theta}{\sqrt{a^2tan^2\theta+a^2}} = \int \frac{asec^2\theta}{a\sec\theta} d\theta = \int \sec\theta\ d\theta = \ln|\sec\theta + \tan\theta| + K$

$= \ln\left|\sqrt{tan^2\theta + 1} + \tan\theta\right| + K = \ln\left|\sqrt{\frac{x^2}{a^2} + 1} + \frac{x}{a}\right| + K$

$= \ln\left|\frac{x+\sqrt{x^2+a^2}}{a}\right| + K$

$= \ln\left|x + \sqrt{x^2 + a^2}\right| + K - \ln|a|$

$= \ln\left|x + \sqrt{x^2 + a^2}\right| + C$ (Where $C = K - \ln|a|$)

5. Let $x = a\sec\theta$, so that $dx = a\sec\theta\tan\theta\ d\theta$

$\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta\ d\theta}{\sqrt{a^2sec^2\theta-a^2}} = \int \frac{a\sec\theta\tan\theta}{a\tan\theta} d\theta = \int \sec\theta\ d\theta$

$= \ln|\sec\theta + \tan\theta| + K = \ln\left|\sec\theta + \sqrt{sec^2\theta - 1}\right| + K$

$= \ln\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + K$

$= \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + K$

$= \ln\left|x + \sqrt{x^2 - a^2}\right| + K - \ln|a|$

$= \ln\left|x + \sqrt{x^2 - a^2}\right| + C$ (Where $C = K - \ln|a|$)

SOME SPECIAL FORMULAE

1. $\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} sin^{-1}\frac{x}{a} + C$

2. $\int \sqrt{x^2 + a^2}dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2 + a^2}\right| + C$

3. $\int \sqrt{x^2 - a^2}dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right| + C$

Solutions:

1. $\int \sqrt{a^2 - x^2}dx = \int 1 \cdot \sqrt{a^2 - x^2}dx$

$= x\sqrt{a^2 - x^2} - \int x\left(\frac{-2x}{2\sqrt{a^2-x^2}}\right)dx$

$= x\sqrt{a^2 - x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}dx$

$= x\sqrt{a^2 - x^2} + \int \frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}}dx$

$= x\sqrt{a^2 - x^2} + a^2\int \frac{dx}{\sqrt{a^2-x^2}} - \int \sqrt{a^2 - x^2}dx$

$\therefore\ 2\int \sqrt{a^2 - x^2}dx = x\sqrt{a^2 - x^2} + a^2\int \frac{dx}{\sqrt{a^2-x^2}}$

$= x\sqrt{a^2 - x^2} + a^2 sin^{-1}\frac{x}{a} + K$

$\therefore\ \int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}sin^{-1}\frac{x}{a} + C$ (Where $C = \frac{K}{2}$)

2. $\int \sqrt{x^2 + a^2}dx = \int 1 \cdot \sqrt{x^2 + a^2}dx$

$= x\sqrt{x^2 + a^2} - \int x\left(\frac{2x}{2\sqrt{x^2+a^2}}\right)dx$

$= x\sqrt{x^2 + a^2} - \int \frac{x^2}{\sqrt{x^2+a^2}}dx$

$= x\sqrt{x^2 + a^2} - \int \frac{(x^2+a^2)-a^2}{\sqrt{x^2+a^2}}dx$

$= x\sqrt{x^2 + a^2} - \int \sqrt{x^2 + a^2}dx + a^2\int \frac{dx}{\sqrt{x^2+a^2}}$

$\therefore\ 2\int \sqrt{x^2 + a^2}dx = x\sqrt{x^2 + a^2} + a^2\int \frac{dx}{\sqrt{x^2+a^2}}$

So, $2\int \sqrt{x^2 + a^2}dx = x\sqrt{x^2 + a^2} + a^2 \ln\left|x + \sqrt{x^2 + a^2}\right| + K$

$\therefore\ \int \sqrt{x^2 + a^2}dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2 + a^2}\right| + C$

(Where $C = \frac{K}{2}$)

3. $\int \sqrt{x^2 - a^2}dx = \int 1 \cdot \sqrt{x^2 - a^2}dx$

$= x\sqrt{x^2 - a^2} - \int x\left(\frac{2x}{2\sqrt{x^2-a^2}}\right)dx$

$= x\sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}dx$

$= x\sqrt{x^2 - a^2} - \int \frac{(x^2-a^2)+a^2}{\sqrt{x^2-a^2}}dx$

$= x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2}dx + a^2\int \frac{dx}{\sqrt{x^2-a^2}}$

$\therefore\ 2\int \sqrt{x^2 - a^2}dx = x\sqrt{x^2 - a^2} - a^2\int \frac{dx}{\sqrt{x^2-a^2}}$

So, $2\int \sqrt{x^2 - a^2}dx = x\sqrt{x^2 - a^2} - a^2 \ln\left|x + \sqrt{x^2 - a^2}\right| + K$

$\therefore\ \int \sqrt{x^2 - a^2}dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right| + C$

(Where $C = \frac{K}{2}$)

METHOD OF INTEGRATION BY PARTIAL FRACTIONS

If the integrand is a proper fraction $\frac{P(x)}{Q(x)}$, then it can be decomposed into simpler partial fractions and each partial fraction can be integrated separately by the methods outlined earlier.

SOME SPECIAL FORMULAE

1. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

2. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

Solutions:

1. We have, $\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$

$\therefore \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int\left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx$

$= \frac{1}{2a}[\ln|x-a| - \ln|x+a|] + C$

$\therefore \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

2. We have, $\frac{1}{a^2-x^2} = \frac{1}{(a+x)(a-x)}$

$= \frac{1}{2a}\left(\frac{1}{a+x} + \frac{1}{a-x}\right)$

$\therefore \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\int\left(\frac{1}{a+x} + \frac{1}{a-x}\right)dx$

$= \frac{1}{2a}[\ln|a+x| - \ln|a-x|] + C$

$\therefore \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

Example:- Evaluate $\int \frac{x^2+1}{(x-1)^2(x+3)}dx$

Solution:- Let $\frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$ ---------------(1)

Multiplying both sides of (1) by $(x-1)^2(x+3)$,

$\Rightarrow \ x^2 + 1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$ ---------------(2)

Putting $x = 1$ in (2),we get,$B = \frac{1}{2}$

Putting $x = -3$ in (2),we get,$10 = 16C \Rightarrow C = \frac{5}{8}$

Equating the co-efficients of x^2 on both sides of (2), we get

$1 = A + C \Rightarrow A = 1 - \frac{5}{8} = \frac{3}{8}$

Substituting the values of A,B &C in (1) ,we get

$\frac{x^2+1}{(x-1)^2(x+3)} = \frac{3}{8}\cdot\frac{1}{x-1} + \frac{1}{2}\cdot\frac{1}{(x-1)^2} + \frac{5}{8}\cdot\frac{1}{x+3}$

$\therefore \int \frac{x^2+1}{(x-1)^2(x+3)}dx$

$= \frac{3}{8}\int\frac{dx}{x-1} + \frac{1}{2}\int\frac{dx}{(x-1)^2} + \frac{5}{8}\int\frac{dx}{x+3}$

$= \frac{3}{8}\ln|x-1| + \frac{5}{8}\ln|x+3| - \frac{1}{2(x-1)} + C$

<u>Example</u>:- Evaluate $\int \frac{x}{(x-1)(x^2+4)} dx$

<u>Solution</u>:- Let $\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$ ---------------(1)

Multiplying both sides of (1) by $(x-1)(x^2+4)$,we get

$x = A(x^2+4) + (Bx+C)(x-1)$--------------(2)

Putting $x = 1$ in (2), we get,$A = \frac{1}{5}$

Putting $x = 0$ in (2), we get,$0 = 4A - C \Longrightarrow C = 4A \Longrightarrow C = \frac{4}{5}$

Equating the co-efficients of x^2 on both sides of (2), we get

$0 = A + B \Rightarrow B = -\frac{1}{5}$

Substituting the values of A, B and C in (1) we get

$$\frac{x}{(x-1)(x^2+4)} = \frac{1}{5(x-1)} - \frac{1}{5}\frac{(x-4)}{(x^2+4)}$$

$\therefore \quad \int \frac{x}{(x-1)(x^2+4)} dx = \frac{1}{5}\int \frac{dx}{x-1} - \frac{1}{5}\int \frac{x-4}{x^2+4} dx$

$= \frac{1}{5}\int \frac{dx}{x-1} - \frac{1}{5}\int \frac{xdx}{x^2+4} + \frac{4}{5}\int \frac{dx}{x^2+4}$

$= \frac{1}{5}\int \frac{dx}{x-1} + \frac{1}{10}\int \frac{2xdx}{x^2+4} + \frac{4}{5}\int \frac{dx}{x^2+4}$

$= \frac{1}{5}\ln|x-1| - \frac{1}{10}\ln|x^2+4| + \frac{2}{5} tan^{-1}\left(\frac{x}{2}\right) + C$

<u>Example</u>:- Evaluate $\int \frac{x^2}{(x^2+1)(x^2+4)} dx$

<u>Solution</u>:- Let $x^2 = y \; Then \frac{x^2}{(x^2+1)(x^2+4)} = \frac{y}{(y+1)(y+4)}$

Let $\frac{y}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4}$ ---------------(1)

Multiplying both sides of (1) by $(y+1)(y+4)$,we get

$y = A(y+4) + B(y+1)$-----------(2)

Putting $y = -1 \; and \; y = -4$ successively in (2),we get,$A = -\frac{1}{3}$ and$B = \frac{4}{3}$

Substituting the values of A and B in (1),we get

$\frac{□}{(□+1)(□+4)} = -\frac{1}{3(□+1)} + \frac{4}{3(□+4)}$

Replacing □ by $□^2$, we obtain

$\frac{□^2}{(□^2+1)(□^2+4)} = -\frac{1}{3(□^2+1)} + \frac{4}{3(□^2+4)}$

$\therefore \quad \int \frac{x^2}{(x^2+1)(x^2+4)} dx = \frac{-1}{3}\int \frac{dx}{x^2+1} + \frac{4}{3}\int \frac{dx}{x^2+4}$

$= -\frac{1}{3} tan^{-1}x + \frac{2}{3} tan^{-1}\left(\frac{x}{2}\right) + C$

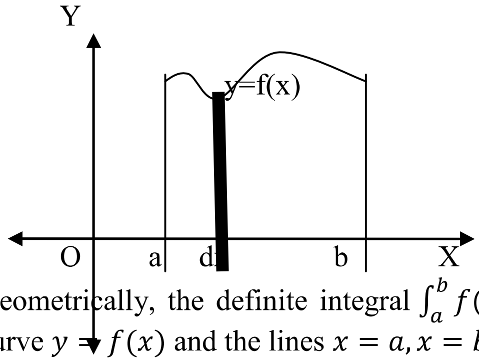
<u>DEFINITE INTEGRAL</u>

If $f(x)$ is a continuous function defined in the interval [a,b] and$F(x)$ is an anti-derivative of $f(x)$ $i.e., \frac{dF(x)}{dx} = f(x)$,then the define integral of $f(x)$ over [a,b] is denoted by

$\int_a^b f(x)dx$ and is equal to $F(b) - F(a)$

$i.e., \int_a^b f(x)dx = F(b) - F(a)$

The constants a and b are called the limits of integration. ‘a’ is called the lower limit and ‘b’ the upper limit of integration.The interval [a,b] is called the interval of integration.



Geometrically, the definite integral $\int_a^b f(x)dx$ is the AREA of the region bounded by the curve $y = f(x)$ and the lines $x = a, x = b$ and x-axis.

EVALUATION OF DEFINITE INTEGRALS

To evaluate the definite integral $\int_a^b f(x)dx$ of a continuous function $f(x)$ defined on [a, b], we use the following steps.

Step-1:-Find the indefinite integral $\int f(x)dx$

Let $\int f(x)dx = F(x)$

Step-2:-Then, we have

$\int_a^b f(x)dx = F(x)]_a^b = F(b) - F(a)$

PROPERTIES OF DEFINITE INTEGRALS

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. $\int_a^b f(x)dx = \int_a^b f(y)dy = \int_a^b f(z)dz$

i.e., definite integral is independent of the symbol of variable of integration.

3. $\int_a^b f(x)dx = \int_a^c f(x)dx = \int_c^b f(x)dx, a < c < b$

4. $\int_0^a f(x)dx = \int_0^a f(a-x)dx, a > 0$

5. $\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & if f(-x)=f(x) \\ 0, & if f(-x)=-f(x) \end{cases}$

6. $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & if f(2a-x)=f(x) \\ 0, & if f(2a-x)=-f(x) \end{cases}$

Example:- Evaluate $\int_0^1 x\tan^{-1}x\, dx$

Solution:-We have, $\int x\tan^{-1}x\,dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{x^2+1}dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{(x^2+1)-1}{x^2+1}dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{dx}{x^2+1}$

$= \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x$

$= \frac{(x^2+1)}{2}\tan^{-1}x - \frac{x}{2}$

$\therefore \int_0^1 x\tan^{-1}x\,dx = \left[\frac{x^2+1}{2}\tan^{-1}x - \frac{x}{2}\right]_0^1$

$= \tan^{-1}1 - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}$

Example:- Evaluate $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx$

Solution:-Let $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx$

$= \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx$

$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx$

$\therefore\ 2I = I + I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx + \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx = \int_0^{\pi/2} \frac{(\sin x + \cos x)}{(\sin x + \cos x)}\,dx$

$= \int_0^{\pi/2} dx = x\Big]_0^{\pi/2} = \frac{\pi}{2}$

$\therefore\ I = \frac{\pi}{4}$

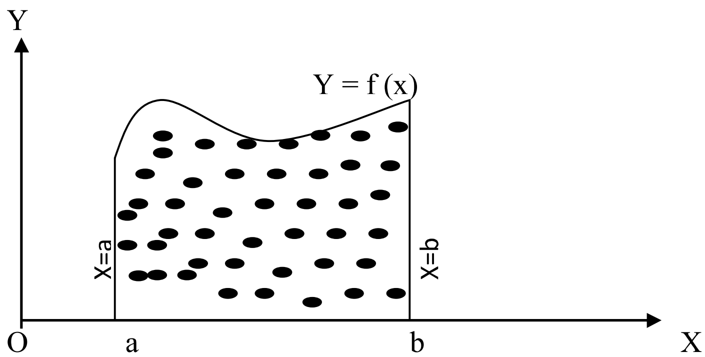
$\therefore\ \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{\pi}{4}$

AREA UNDER PLANE CURVES

DEFINITION-1:-

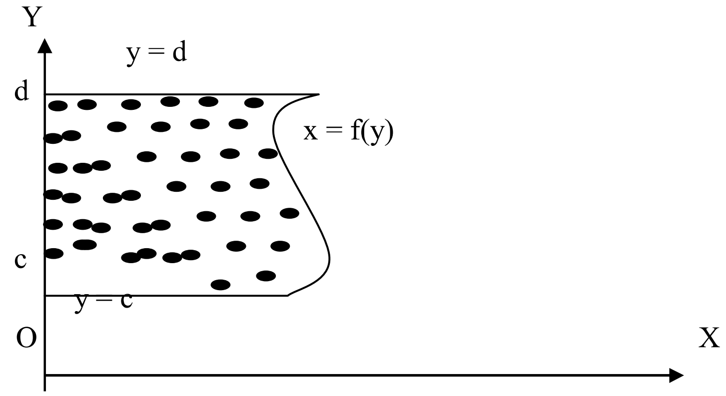
Area of the region bounded by the curve $y = f(x)$, the X-axis and the lines $x = a, x = b$ is defined by

Area $= \left|\int_a^b y\,dx\right| = \left|\int_a^b f(x)\,dx\right|$



DEFINITION-2:-Area of the region bounded by the curve $x = f(y)$, the Y-axis and the lines $y = c, y = d$ is defined by

Area=$\left|\int_c^d x dy\right| = \left|\int_c^d f(y) dy\right|$



Example:-Find the area of the region bounded by the curve $y = e^{3x}$, x-axis and the lines $x = 4, x = 2$.

Solution:-The required area is defined by

$$A=\int_2^4 e^{3x} dx = \frac{1}{3} e^{3x}\Big]_2^4 = \frac{1}{3}(e^{12x} - e^{6x})$$

Example:-Find the area of the region bounded by the curve $xy = a^2$, y-axis and the lines $y = 2, y = 3$

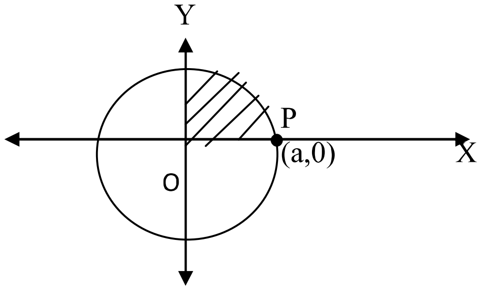
Solution:- $We\ have,\ xy = a^2 \Rightarrow x = \frac{a^2}{y}$

∴ The required area is defined by

$$A=\int_2^3 x dy = a^2 \int_2^3 \frac{dy}{y} = [a^2 \ln y]_2^3 = a^2(\ln 3 - \ln 2) = a^2 \ln\left(\frac{3}{2}\right)$$

Example:-Find the area of the circle $x^2+y^2 = a^2$

Solution:-We observe that, $y = \sqrt{a^2-x^2}$ in the first quadrant.



∴ The area of the circle in the first quadrant is defined by,

$A_1 = \int_0^a \sqrt{a^2-x^2}\, dx$

As the circle is symmetrically situated about both X −axis and Y −axis, the area of the circle is defined by,

A=4 $\int_0^a \sqrt{a^2-x^2}\,dx$

=4 $\left|\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}sin^{-1}\frac{x}{a}\right|_0^a$

$= 4\frac{a^2}{2}sin^{-1}1 = 2a^2\frac{\pi}{2} = \pi a^2.$

DIFFERENTIAL EQUATIONS

DEFINITION:-An equation containing an independent variable (x), dependent variable (y) and differential co-efficients of dependent variable with respect to independent variable is called a differential equation.

For distance,

1. $\frac{dy}{dx} = \sin x + \cos x$
2. $\frac{dy}{dx} + 2xy = x^3$
3. $y = x\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Are examples of differential equations.

ORDER OF A DIFFERENTIAL EQUATION

The order of a differential equation is the order of the highest order derivative appearing in the equation.

Example:-In the equation, $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^x$,

The order of highest order derivative is 2. So, it is a differential equation of order 2.

DEGREE OF A DIFFERENTIAL EQUATION

The degree of a differential equation is the integral power of the highest order derivative occurring in the differential equation, after the equation has been expressed in a form free from radicals and fractions.

Example:-Consider the differential equation $\frac{d^3y}{dx^3} - 6\left(\frac{dy}{dx}\right)^2 - 4y = 0$

In this equation the power of highest order derivative is 1.So, it is a differential equation of degree 1.

Example:-Find the order and degree of the differential equation

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = K\frac{d^2y}{dx^2}$$

Solution:- By squaring both sides, the given differential equation can be written as $K^2\left(\frac{d^2y}{dx^2}\right)^2 - \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = 0$

The order of highest order derivative is 2.So, its order is 2.
Also, the power of the highest order derivative is 2.So, its degree is 2.

FORMATION OF A DIFFERENTIAL EQUATION

An ordinary differential equation is formed by eliminating certain arbitrary constants from a relation in the independent variable, dependent variable and constants.

Example:-Form the differential equation of the family of curves $y = a\sin(bx + c)$, a and c being parameters.

Solution:-We have $y = a\sin(bx + c)$ -------------(1)

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = ab\cos(bx + c) \quad \text{----------------------(2)}$$

Differentiating (2) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -ab^2\sin(bx + c) \quad \text{-------------------(3)}$$

Using (1) and (3), we get

$$\frac{d^2y}{dx^2} = -b^2y$$

$$\therefore \quad \frac{d^2y}{dx^2} + b^2y = 0$$

This is the required differential equation.

Example:-Form the differential equation by eliminating the arbitrary constant in $y = Atan^{-1}x$.

Solution:-We have, $y = Atan^{-1}x$ ------------------(1)

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{A}{1+x^2} \quad \text{-------------------(2)}$$

Using (1) and (2), we get

$$\frac{dy}{dx} = \frac{y}{(1+x^2)tan^{-1}x}$$

$$\therefore \ (1 + x^2)tan^{-1}x\frac{dy}{dx} = y$$

This is the required differential equation.

SOLUTION OF A DIFFERENTIAL EQUATION

A solution of a differential equation is a relation $(like\ y = f(x)\ or\ f(x,y) = 0)$between the variables which satisfies the given differential equation.

GENERAL SOLUTION

The general solution of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation.

PARTICULAR SOLUTION

A particular solution is that which can be obtained from the general solution by giving particular values to the arbitrary constants.

SOLUTION OF FIRST ORDDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS

We shall discuss some special methods to obtain the general solution of a first order and first degree differential equation.

1. Separation of variables

2. Linear Differential Equations

3. Exact Differential Equations

SEPARATION OF VARIABLES

If in a first order and first degree differential equation, it is possible to separate all functions of x and dx on one side, and all functions of y and dy on the other side of the equation, then the variables are said to be separable.Thus the general form of such an equation is $f(y)dy = g(x)dx$
Then, Integrating both sides, we get

$\int f(y)dy = \int g(x)dx + C$ as its solution.

Example:-Obtain the general solution of the differential equation

$$9y\frac{dy}{dx} + 4x = 0$$

Solution:- We have, $9y\frac{dy}{dx} + 4x = 0$

$$\Rightarrow 9y\frac{dy}{dx} = -4x$$

$$\Rightarrow 9y\, dy = -4x\, dx$$

Integrating both sides, we get

$$9\int ydy = -4\int xdx$$

$$\Rightarrow \frac{9}{2}\cdot y^2 = \frac{-4}{2}x^2 + K$$

$$\Rightarrow 9y^2 = -4x^2 + C \quad \text{(Where C=2K)}$$

$$\Rightarrow 4x^2 + 9y^2 = C$$

This is the required solution

LINEAR DIFFERENTIAL EQUATIONS

A differential equation is said to be linear, if the dependent variable and its differential co-efficients occurring in the equation are of first degree only and are not multiplied together.

The general form of a linear differential equation of the first order is

$$\frac{dy}{dx} + Py = Q, \quad \text{-----------------(1)}$$

Where P and Q are functions of x.

To solve linear differential equation of the form (1),
at first find the Integrating factor = $e^{\int P\,dx}$ -------------------(2)

It is important to remember that

I.F = $e^{\int P \cdot dx}$

Then, the general solution of the differential equation (1) is

$$y.(I.F) = \int Q.(I.F)dx + C \quad \text{-------------------(3)}$$

Example:-Solve $\frac{dy}{dx} + y\sec x = \tan x$

Solution:-The given differential equation is

$$\frac{dy}{dx} + (\sec x)y = \tan x \quad \text{-----------------(1)}$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \text{ , where } P = \sec x \text{ and } Q = \tan x$$

∴ I.F= $e^{\int P\cdot dx} = e^{\int \sec x dx} = e^{\ln(\sec x + \tan x)}$

So, I.F=$\sec x + \tan x$

∴ The general solution of the equation (1) is

$$y.(I.F) = \int Q(I.F)dx + C$$

$$\Rightarrow \quad y\,(\sec x + \tan x) = \int \tan x\,(\sec x + \tan x)dx + C$$

$$\Rightarrow \quad y\,(\sec x + \tan x) = \int(\tan x \ \sec x + tan^2 x)dx + C$$

$$\Rightarrow \quad y\,(\sec x + \tan x) = \int(\tan x \ \sec x + sec^2 x - 1)dx + C$$

$$\Rightarrow \quad y\,(\sec x + \tan x) = \sec x + \tan x - x + C$$

This is the required solution.

Example:-Solve:$(1 + x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$

Solution:-The given differential equation can be written as

$$(1+x^2)\frac{dy}{dx}+2xy=4x^2$$

$$\Rightarrow \frac{dy}{dx}+\frac{2x}{1+x^2}\cdot y=\frac{4x^2}{1+x^2} \quad \text{----------------(1)}$$

This is a linear equation of the form $\frac{dy}{dx}+Py=Q$,

Where $P=\frac{2x}{1+x^2}$ and $Q=\frac{4x^2}{1+x^2}$

We have, I.F = $e^{\int P\cdot dx}=e^{\int 2x/(1+x^2)dx}=e^{\ln(1+x^2)}=1+x^2$ ---------------(2)

∴The general solution of the given differential equation (1) is

$$y.(I.F)=\int Q.(I.F)dx+C$$

$$\Rightarrow \quad y(1+x^2)=\int \frac{4x^2}{1+x^2}.(1+x^2)dx+C$$

$$\Rightarrow \quad y(1+x^2)=4\int x^2dx+C$$

$$\Rightarrow \quad y(1+x^2)=\frac{4}{3}x^3+C$$

This is the required solution

EXACT DIFFERENTIAL EQUATIONS

DEFINITION:- A differential equation of the form $M(x,y)dx+N(x,y)dy=0$ is said to be exact if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

METHOD OF SOLUTION:-

The general solution of an exact differentia equation $Mdx+Ndy=0$ is

$$\int Mdx+\int (terms\ of\ N\ not\ containing\ x)dy=C,$$

(y=constant)

Provided $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Example:- Solve;$(x^2-4xy-2y^2)dx+(y^2-4xy-2x^2)dy=0$.

Solution:-The given differential equation is of the form $Mdx+Ndy=0$.

Where, $M=x^2-4xy-2y^2$ and $N=y^2-4xy-2x^2$

We have $\frac{\partial M}{\partial y}=-4x-4y=\frac{\partial N}{\partial x}$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, so the given differential equation is exact.

∴ The general solution of the given exact differential equation is

$$\int Mdx+\int (terms\ of\ N\ free\ from\ x)dy=C$$

(y=constant)

$$\Rightarrow \quad \int (x^2-4xy-2y^2)dx+\int y^2dy=C$$

(y=constant)

$\Rightarrow \quad \frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} = C$
$\Rightarrow \quad x^3 - 6x^2y - 6xy^2 + y^3 = C.$
This is the required solution.

Example:- Solve;$(x^2 - ay)dx = (ax - y^2)dy$

Solution:-The given differential equation can be written as

$(x^2 - ay)dx + (y^2 - ax)dy = 0$ ----------------(1)
Which is of the form $Mdx + Ndy = 0,$
Where,$M = x^2 - ay$ and $N = y^2 - ax.$
We have $\frac{\partial M}{\partial y} = -a \text{ and } \frac{\partial N}{\partial x} = -a$
Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation (1) is exact.
∴ The solution of (1) is $\int (x^2 - ay)dx + \int y^2 dy = C$
(y=constant)
$\Rightarrow \frac{x^3}{3} - axy + \frac{y^3}{3} = C$
$\Rightarrow x^3 - 3axy + y^3 = C,$
Which is the required solution.

Example:- Solve; $ye^{xy}dx + (xe^{xy} + 2y)dy = 0.$

Solution:- The given differential equation is $ye^{xy}dx + (xe^{xy} + 2y)dy = 0,$

Which is of the form $Mdx + Ndy = 0.$
Where,$M = ye^{xy}$ and $N = xe^{xy} + 2y$
We have $\frac{\partial M}{\partial y} = e^{xy} + xye^{xy} = \frac{\partial N}{\partial x}$
So the given equation is exact and its solution is
$\int ye^{xy}dx + \int 2ydy = C.$
(y=constant)
$\Rightarrow e^{xy} + y^2 = C$

Example:- Solve; $(3x^2 + 6xy^2)dx + (6x^2y + 4y^3)dy = 0$

Solution:- The given equation is of the form $Mdx + Ndy = 0,$
Where,$M = 3x^2 + 6xy^2$ and $N = 6x^2y + 4y^3$
We have $\frac{\partial M}{\partial y} = 12xy = \frac{\partial N}{\partial x}.$
So the given equation is exact and its solution is
$\int (3x^2 + 6xy^2)dx + \int (4y^3)dy = C$
(y=constant)
$\Rightarrow \frac{3x^3}{3} + \frac{6}{2}x^2y^2 + \frac{4}{4}y^4 = C$

$\Rightarrow x^3 + 3x^2y^2 + y^4 = C$
This is the required solution.